Let us consider the interval $[u_m = 0, u_M]$ where the minimization in (11) is desired, and suppose that it is narrow enough to make it possible that both $F(u,\varphi)$ and $\hat{F}(u)$, defined in Section III.B, can be accurately approximated by the first three terms of their Taylor expansion (around the direction of maximum radiation, that is, $u = 0$). Accordingly, the minimization in (11) can be achieved by constraining $F(u,\varphi)$ and $\hat{F}(u)$, along with their first and second derivatives, to be equal in $u = 0$. By doing so, we easily obtain condition (12). We only note that the expression of $F(u,\varphi)$ can be obtained by simplifying (14) to the case of one single ring, whereas $\hat{F}(u)$ comes by restricting to the interval $[r_i, r_e]$ the integration domain of the function $\hat{f}(u)$ recalled above.

We stress that enforcement of the condition involving the second derivatives (not used in the 1D DA in [14]) is necessary in this 2D case because the first derivatives of $F(u,\varphi)$ and $\hat{F}(u)$ are identically equal to zero in $u = 0$. Moreover, by relaxing the approximation in (14) an additional term (proportional, through the factor $2/\beta^2$, to the second derivative, in $u = 0$, of the element pattern) would appear at the right hand side of (12). However, such an additional term turned out to play a negligible role in the performances of the final array, thus confirming that the approximation in (14) is very sound in our case of interest.

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A Note on the Construction of Synthetic Basis Functions for Antenna Arrays

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***Abstract*—Construction of synthetic basis functions (SBFs) for analyzing the radiation problems of antenna arrays is discussed. The SBFs of a block are generated from two separate solution spaces, with one space to account for the effect of the feeding voltages, and the other space to account for the responses of the block to incident electromagnetic fields from its environment. Singular value decomposition (SVD) method is used to extract the characteristic modes of the block. Examples show that using SBFs constructed in this way may cause better efficiency.**

***Index Terms*—Domain decomposition method, singular-value decomposition, synthetic basis function.**

I. INTRODUCTION

The rigorous analysis of radiation problems involving large antenna arrays becomes possible [1], [2] owing to the development of innovative numerical techniques in computational electromagnetics. The target of many of these techniques is to reduce the severe computational cost imposed by the method of moments (MoM) [3] when analyzing electrically large problems. In the conventional implementation of MoM, a typical spatial sampling rate of $\lambda/10$ is usually required for discretization if low order basis functions, such as Rao-Wilton-Glisson (RWG) basis functions for surfaces [4] or piece-wire linear functions for wires, are used to expand the induced currents. The direct solution of the linear matrix system is very expensive for electrically large problems. Some approaches to overcome this difficulty have been proposed, of which the most important one is the fast multipole method (FMM) [5] and its extension—the multilevel fast multipole algorithm (MLFMA). In these methods, only the near-field terms of the coupling matrix need to be stored and fast matrix-vector production in the iteration process is achieved by performing factorization of the Green's function. The computational complexity is reduced to $O(N \log N)$ in the multilevel fast multipole algorithm. A lot of techniques utilize this efficient matrix-vector production method in the iteration solution of large systems, such as the complex mutipole beam approach (CMBA) [6], the impedance matrix localization (IML) technique [7], the adaptive method (AIM) [8] or the multilevel matrix decomposition algorithm (MLMDA) [9].

Another branch of methodology for rigorously analyzing large scale system focuses on reducing the number of unknowns involved in the final linear system. Generally speaking, much less unknowns are required to solve in methods adopting entire domain basis functions than in methods adopting subdomain basis functions. However, entire domain basis functions are conventionally continuous eigen functions corresponding to some kind of boundary value problems of the concerned system, which are usually difficult to solve for complicated system. A natural consideration is to find a kind of basis functions that

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bear the following features: (1) they can efficiently express the electromagnetic characteristics of the system; (2) they are easy to obtain in a routine way. Apparently, basis functions defined on middle-sized domains may be good choices. Usually, these kinds of basis functions can be aggregated from low order basis functions, such as RWGs. It is possible that two domains may share a common ground plane, substrates, or may be connected by thin PEC strips. In these cases, half basis functions, such as half RWGs (HRWGs), have to be used for those bases that are cut apart by the interface between the two domains. Each middle-sized domain basis function corresponds to a characteristic solution of the concerned domain under typical electromagnetic situations. The methods using macro-basis functions (MBFs), characteristic basis functions (CBFs) [11] and synthetic basis functions (SBFs) [10], [12] belong to this category. The SBF method is perhaps more versatile since each SBF, obtained by SVD process, is actually a discrete eigen function of a solution space to the concerned domain, which can be regarded as a block-wise entire domain basis function expressed by low order basis functions. In SBF method, if the solution space is properly chosen, a very small number of SBFs are enough to describe the electromagnetic characteristics of the domain. Therefore, high compression rate of number of unknowns is achievable.

In this paper, the effect of using different solution space is discussed. We use antenna arrays as numerical examples to show that better efficiency may be achievable if we construct the SBFs using more adequately defined solution spaces.

II. Construction of SBFs

The radiation problem of a PEC antenna occupying a surface S is often analyzed based on the electric field integral equation (EFIE),

$$j\omega\mu \int_S \left[\vec{J} + \frac{1}{k^2}\nabla(\nabla\cdot\vec{J})gdS\right]\Bigg|_{\tan} = \vec{E}_0(\vec{r})|_{\tan} \quad (1)$$

which can be compactly written as

$$\mathcal{L}\{J(\vec{r})\} = E_0(\vec{r})|_{\tan}, \quad \vec{r}\in S \quad (2)$$

where $\mathcal{L}$ is a linear operator and $E_0(\vec{r})$ is the imposed electric field. The EFIE is discretized to get a linear system

$$[Z][I] = [V]. \quad (3)$$

Consider an antenna array with M elements. We use N_b low order basis functions to expand the current distribution on each element. Therefore, the excitation vector $[V]$ and the current coefficient vector $[I]$ are both MN_b column vectors. The size of the impedance matrix is $MN_b \times MN_b$, with entries calculated by inner products

$$Z_{p,q} = \langle \vec{f}_p, \mathcal{L}\{\vec{f}_q\}\rangle \quad (4)$$

where $\vec{f}_p$ is the pth basis function, and $p,q = 1,2,\ldots,MN_b$.

Apply domain decomposition method to antenna array shown in Fig. 1, where each element is fed independently at one or several feeding points. The current distribution in the mth element can be written as

$$\left[I^{(m)}\right] = \left[Z^{(m,m)}\right]^{-1}\left[V^{(m)}\right] - \sum_{n=1,n\neq m}^{N_b}\left[Z^{(m,m)}\right]^{-1}\left[Z^{(m,n)}\right]\left[I^{(n)}\right] \quad (5)$$

where $[I^{(m)}]$, $[V^{(m)}]$ and $[Z^{(m,m)}]$ are respectively the current distribution vector, the feeding vector and the self-impedance matrix of the mth element. The matrix $[Z^{(m,n)}]$ is the mutual coupling impedance matrix between the mth element and the nth element. From (5) we can see clearly that two factors have influences on the current distribution of an element: the feeding voltages on the element and the coupling fields from the other elements. The effect due to these two factors may possibly be different. Instead of constructing one single solution space to include the effect of these two factors, as was proposed in [10], in this paper we define two separate solution spaces, one space for the feeding voltages, the other space for the mutual coupling fields. The SBFs are found from each solution space using SVD process separately.

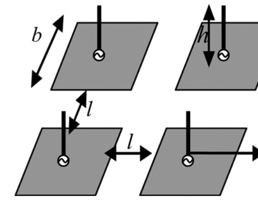

Fig. 1. A PEC antenna array with elements fed with independent feeding voltages. Each element contains a straight wire perpendicularly mounted on a $b\times b$ PEC square.

The first solution space $[R_1]$ is used to contain the radiation fields caused by all internal sources. It can be created in a way similar to that discussed in [10], which is denoted by

$$[R_1] = \left[\left[r^{(1)}\right], \left[r^{(2)}\right], \ldots, \left[r^{(N_S)}\right]\right]. \quad (6)$$

The column $[r^{(u)}]$ is the current distribution on the element when it is isolated and is fed by the uth feeding voltage alone. Applying SVD to (6) yields

$$[R_1] = [U_1][\rho_1][W_1]^T. \quad (7)$$

The columns of $[U_1]$ corresponding to those singular values larger than a properly chosen threshold are used as SBFs for the first solution space. In our example shown in Fig. 1, only one feeding voltage exists in an element. Therefore, the first solution space contains only one column. It is not necessary to perform SVD to it. We can simply use the sole column vector as the SBF to account for the effect of this solution space. For a block containing many interior sources, or distributed sources, SVD process is required to extract the SBFs.

The second solution space $[R_2]$ is used to contain the scattered fields caused by incident fields from the environments. It is constructed exactly following the way described in [10], however, the auxiliary sources are arranged with the method proposed in [12]. The SBFs for this space can also be found using SVD method. The total SBFs for this element include the SBFs from the two spaces. Each SBF is an aggregation of the low order basis functions. We denote the kth SBF of the mth element as

$$\vec{P}_k^m = \sum_{i=1}^{N_b} U_k^{(m)}(i)\vec{f}_i^{(m)}, \quad k = 1,2,\ldots M_b \quad (8)$$

where $\vec{f}_i^{(m)}$ is the ith low order basis function on the mth element, $U_k^{(m)}(i)$ is the kth column vector of the matrix $[U]$, M_b is the number of the SBFs on the element. There are totally MM_b SBFs for the whole PEC antenna array.

The current distribution can then be expressed by SBFs, i.e.,

$$\vec{J} = \sum_{m=1}^{M}\sum_{k=1}^{M_b}\varphi_k^{(m)}\vec{P}_k^{(m)}. \quad (9)$$

Substituting (9) into (2) and testing it with SBFs $\vec{P}_l^{(n)*}$ yields

$$\left[A_{n,l}^{m,k}\right]\left[\varphi_k^{(m)}\right] = \left[U_l^{(n)*}\right]^t \cdot \left[V^{(m)}\right] \quad (10)$$

where the upper script t stands for transpose and the asterisk for conjugate. The entries of the size-reduced coefficient matrix $[A_{n,l}^{m,k}]$ can be calculated with the self- and mutual-impedance matrices as follows:

$$\left[A_{n,l}^{m,k}\right] = \left[U_l^{(n)*}\right]^t \cdot \left[Z^{(m,n)}\right] \cdot \left[U_k^{(m)}\right]. \tag{11}$$

The number of SBFs required is usually much less than that of the low order basis function, the size of (10) is largely reduced and in many cases it can be solved directly.

III. Numerical Results

Three PEC antenna arrays are checked to show the effect of how the solution spaces are constructed. The element of the first array has a straight PEC wire placed perpendicularly to a PEC plate, as shown in Fig. 1. The feeding voltages are imposed at the wire-plate junctions. The working frequency is 0.3 GHz. The geometrical structure of the array is illustrated in Fig. 1, where $a = 0.5$ m, $b = 0.914$ m, $d = 0.1$ m, $l = 0.2$ m, $u = 0.1$ m, $h = 0.421$ m.

For each element, 375 RWGs are used to mesh the PEC plate, and 14 roof-top basis functions are used for the straight wire, together with a wire-plate junction basis function [13]. As a result, 1560 basis functions are used for the array.

The array is analyzed using SBFs constructed with the proposed procedure in this note and that proposed in [10]. We will measure the accuracy of the SBF methods with respect to that obtained by rigorous MoM algorithm, using the relative errors of the values $(|rE(\theta_i, 0)|)$ as follows

$$Err = \left(\frac{\sum_{i=1}^{180} \left| rE(\theta_i, 0)|_{\mathrm{SBF}} - rE(\theta_i, 0)|_{\mathrm{MoM}} \right|^2}{\sum_{i=1}^{180} \left| rE(\theta_i, 0) \right|_{\mathrm{MoM}}^2} \right)^{0.5} \times 100\% \tag{12}$$

where $\theta_i = i\pi/180$, $i = 1, 2, \ldots, 180$, and $E(\theta_i, 0)$ is the electric field value at the point $(r, \theta_i, 0)$.

The values of Err are obtained by using SBFs that are generated with the proposed method in this note and that proposed in [10]. The calculated results of Err with $1 \leq N_{\mathrm{SBF}} \leq 25$ are shown in Fig. 2, where N_{SBF} is the number of SBFs on a single block.

As shown in Fig. 2, the Err obtained in this paper is always less than that using the method in [10] with the same N_{SBF}. With 25 SBFs in one element, the value of Err with the new method is about 1.4%, while that with the method in [10] is 5.6%.

The radiation fields are plotted in Fig. 3. The result with $N_{\mathrm{SBF}} = 12$ is very close to that by using MoM, while the unknown number is only 4% of that in MoM. However, when the method in [10] is used, the result with $N_{\mathrm{SBF}} = 25$ is still not satisfactory, as can be seen from Fig. 3.

The second example is a phased array with 9 units placed uniformly along a circle with radius of $R = \lambda/4$. Each unit is a bow-tie antenna with geometrical parameters shown in Fig. 4, with A being the feeding point. The phased array is assumed to be set up around a PEC pole with square cross section. The working frequency is 2.4 GHz. In order to evaluate the effect of the pole, the phased array is treated as a single block. The surface current on the array is expanded with 142×9 RWGs, and that on the pole is expanded with 316 RWGs. The auxiliary sources are put on the surface of a hollow cylinder with circumferences indicated by the dashed lines in Fig. 4. With this arrangement, the second solution space contains not only the scattered fields caused by the incident fields coming from the exterior side of the antenna array circle but also those caused by the incident fields coming from the interior side of the antenna array circle, e.g., the fields scattered by the pole. The first solution space contains 9 vectors. Each vector corresponds to the current distribution by feeding the array one

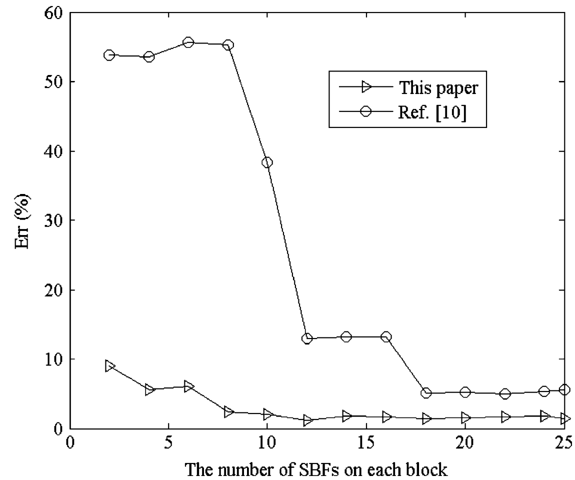

Fig. 2. The Err versus the number of SBFs on each block in the second antenna array. "This paper": using the SBF method in this paper; " [10]": using the SBF method in [10].

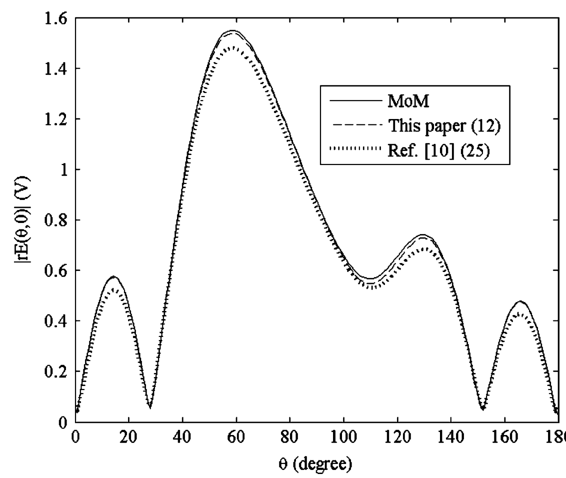

Fig. 3. The calculated $|rE(\theta, 0)|$ of the first antenna array. "This paper (12)": using the method in this paper with 12 SBFs on each block. " [10] (25)": using the method in [10] with 25 SBFs on each block.

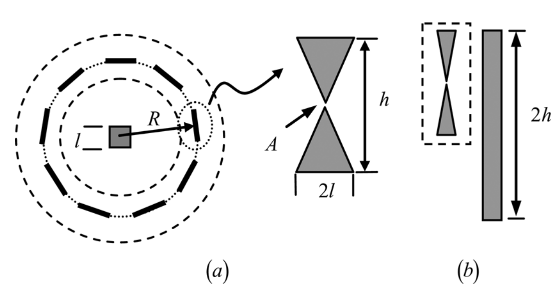

Fig. 4. A phased array placed around a PEC pole. $h = 20$ mm, $l = 16$ mm. (a) Top view. (b) A part of the side view.

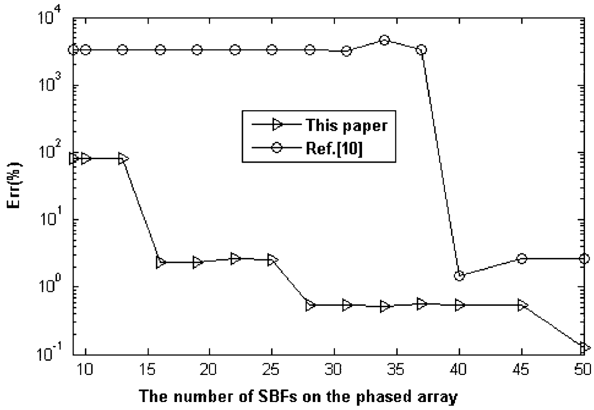

Fig. 5. The Err versus the number of SBFs on the phased array in Fig. 4. "This paper": using the SBF method in this paper; " [10]": using the SBF method in [10].

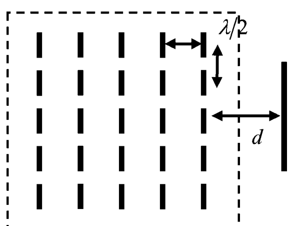

Fig. 6. A 5 × 5 phased array near a PEC plate.

unit at a time. Following the proposed procedure, SVD process is applied to get the SBFs of the two spaces separately. It is found that in this case, the SBFs of the first solution space are almost exactly the 9 vectors that formed the space.

Equation (12) is used to evaluate the effect of the method. In the case that all feeding voltages are of the same phase, the numerical errors using the proposed method and the conventional one are plotted in Fig. 5. With the same number of SBFs, much better accuracy can be achieved using the proposed method.

The third example is a 5 × 5 phased array with the same bow-tie unit placed uniformly with center separation of $\lambda/2$, as shown in Fig. 4. A $2\lambda \times 2\lambda$ PEC plate is placed near the array to serve as an obstacle. The array is also treated as a single block, and the SBFs are generated using auxiliary sources on the surface of a cuboid containing the array, as depicted by the dashed line in Fig. 6. In this case, the first solution space has 25 vectors. It can be found that the eigen functions extracted using SVD procedure are still very similar to the original vectors. This is not strange since the 25 vectors are highly independent. Therefore, we can use the original 25 vectors as SBFs directly.

The surface current on the array is expanded with 142 × 25 RWGs, and that on the plate is expanded with 296 RWGs. The numerical errors associated with the two methods of SBF generation are plotted in Fig. 7, where the array is supposed to be a broadside array, i.e., all feeding voltages are of the same phase. Again, the proposed method shows advantage over the conventional method.

The effects due to the following variations are also checked in this example: (1) the distance d between the array and the PEC plate; (2) the phase of array unit. The SBFs are needed to be generated only once, and can be reused to evaluate the influence of all these variations. We have compared the behavior of the proposed method and the conventional one. It is verified that, at least in all cases we have checked, to get the same level of accuracy, much fewer SBFs are needed if the proposed method is used. Because all the results of Err are similar to the plots in Fig. 7, they are not presented here.

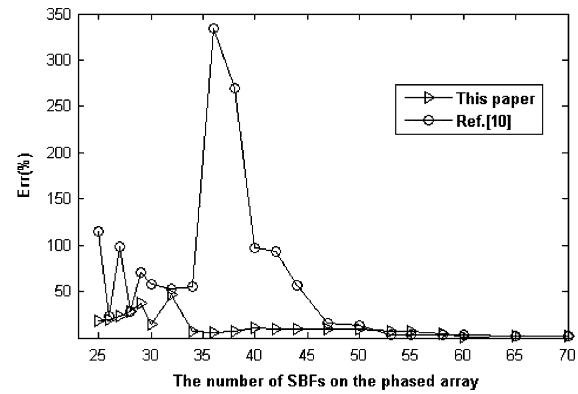

Fig. 7. The Err versus the number of SBFs on the phased array in Fig. 6, with $d = 2\lambda$. "This paper": using the SBF method in this paper; " [10]": using the SBF method in [10].

IV. Conclusion

In many situations such as analyzing antenna arrays, the electromagnetic characteristics of a block is affected by some different factors. Therefore, it is more reasonable to construct SBFs of the block from several separate solution spaces than to construct them from a single solution space. In this way, the electromagnetic characteristics of the block may possibly be expressed more efficiently by SBFs, which will lead to a higher compression rate of the number of unknowns in solving the electromagnetic problems of these systems.

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